Time-reversal and parity conservation for gravitating quarks

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Abstract

The complex mass term of a quark does not violate time-reversal or parity in gravitational interactions, in spite of an axial anomaly.

Time-reversal is known to be violated in the weak interactions. Mass terms $\bar{q}_L M q_R + \bar{\tilde{q}}_L \tilde{M} \tilde{q}_R$+hc are generated for quarks q having charge $+\frac{2}{3}$ and quarks $\tilde{q}$ having charge $-\frac{1}{3}$ in the shape of complex matrices $M, \tilde{M}$ from electroweak symmetry breaking. On diagonalization of $M, \tilde{M}$ through flavour matrices,

$$q_L \to A_L^{-1} q_L, \qquad q_R \to A_R^{-1} q_R \tag{1}$$

$$\tilde{q}_L \to \tilde{A}_L^{-1} \tilde{q}_L, \qquad \tilde{q}_R \to \tilde{A}_R^{-1} \tilde{q}_R, \tag{2}$$

charged current weak interactions pick up a matrix $A_L \tilde{A}_L^{-1} \equiv C$, the Cabibbo-Kobayashi-Maskawa matrix, which is complex and violates time-reversal. The diagonalized mass terms too may be complex. One can write

$$\begin{aligned} \bar{\psi}_L m e^{i\theta} \psi_R + hc &= \bar{\psi} m e^{i\theta\gamma_5} \psi \\ &= \cos\theta \bar{\psi} m \psi + i \sin\theta \bar{\psi} m \gamma_5 \psi. \end{aligned} \tag{3}$$

This looks like a combination of a scalar and a pseudoscalar, suggesting parity violation. More important, the phase factor $e^{i\theta} \to e^{-i\theta}$ under antilinear operations, suggesting additional time-reversal violation in the standard model beyond the weak interaction violation mentioned above.

A chiral transformation may be used to remove θ:

$$\begin{aligned} \psi &\to e^{-i\theta\gamma_5/2} \psi, \\ \bar{\psi} &\to \bar{\psi} e^{-i\theta\gamma_5/2}. \end{aligned} \tag{4}$$

θ does get removed from the mass term, but may reappear as a topological term in the gluon sector [1] through the Jacobian of the fermion measure which is not invariant under the chiral transformation. This is because of the axial

anomaly. However, the question of time-reversal violation depends more subtly on the fermion measure and is avoided in quantum chromodynamics with an appropriate choice of measure [2].

What happens if one considers the gravitational coupling of the quarks? In a gravitational field too, there is an axial anomaly [3] given by

$$\frac{\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\sigma\tau} R_{\alpha\beta}{}^{\sigma\tau}}{384\pi^2}. \tag{5}$$

Time-reversal and parity violations could show up in gravitational interactions of quarks in the same way that strong interactions had been thought to involve these symmetry violations. This is the question we analyze here.

If there is a complex fermion mass term in curved spacetime, its phase θ may again be removed from the fermion action by a chiral transformation, so that there is no direct parity or time-reversal violation, suggesting a redefinition of parity and time-reversal at least at the *classical* level. Whether there is any indirect violation of the symmetries through $R\tilde{R}$ terms generated by the chiral transformation will be investigated below through the fermion determinant.

The Dirac operator without any mass term may be written as

$$\not{D} = \gamma^l e_l^\mu (\partial_\mu - iA_\mu - \frac{i}{2} A_\mu^{mn} \sigma_{mn}). \tag{6}$$

Here, e_l^μ is the tetrad, A_μ^{mn} is the spin connection, and $\sigma_{mn} \equiv i[\gamma_m, \gamma_n]$. A_μ is any other gauge field coupling vectorially to the fermion.

For a real mass term, the fermion action is

$$\int \bar{\psi}(i \not{D} - m)\psi. \tag{7}$$

This is invariant under the standard parity transformation

$$\begin{aligned} \psi(\vec{x}) &\rightarrow \gamma^0 \psi(-\vec{x}), & (8) \\ e_l^\mu(\vec{x}) &\rightarrow \pm e_l^\mu(-\vec{x}), & (9) \end{aligned}$$

where the $\pm$ sign is negative if an odd number of Greek or Latin spatial indices is involved. A_μ^{mn} also changes, but it is determined by e_l^μ. Note that the invariance involves more than the usual flat spacetime symmetry: the σ_{mn} matrices produce $\pm$ signs when γ^0 is taken across it, but these exactly cancel with the $\pm$ signs from any Latin spatial indices of A_μ^{mn}.

Similarly, the action is also invariant under the time-reversal transformation

$$\begin{aligned} \psi(x^0) &\rightarrow i\gamma^1\gamma^3 \psi(-x^0), & (10) \\ e_l^\mu(x^0) &\rightarrow \pm e_l^\mu(-x^0), & (11) \end{aligned}$$

where the $\pm$ sign is negative if an odd number of Greek or Latin temporal indices is involved. Once again the σ_{mn} matrices produce $\pm$ signs which cancel $\pm$ signs from Latin temporal indices of A_μ^{mn}.

The fermion action with a *complex* mass term is

$$\int \bar{\psi}(i \not{D} - m \exp(i\theta\gamma^5))\psi. \tag{12}$$

The mass term involves the matrix $\gamma^0 \exp(i\theta\gamma^5)$ and it is not difficult to guess that the action will be invariant under the chirally rotated parity transformation

$$\psi(\vec{x}) \to \gamma^0 \exp(i\theta\gamma^5)\psi(-\vec{x}). \tag{13}$$

The chiral phase factor $\exp(i\theta\gamma^5)$ commutes with σ_{mn}, which does not therefore provide any additional complication.

Like the parity transformation, the action is also invariant under a chirally rotated time-reversal transformation

$$\psi(x^0) \to i \exp(i\theta\gamma^5)\gamma^1\gamma^3\psi(-x^0). \tag{14}$$

These discrete symmetries have been seen so far at the classical level. The symmetries could break down at the quantum level, *i.e.,* they could be anomalous. There is in fact a reason to suspect such an eventuality because the symmetry transformations involve chiral rotations, and chiral transformations are known to develop anomalies upon quantization. The question here is whether there exist ways of regularizing the theory so that the classical *discrete* symmetries are preserved.

Let us use a zeta function regularization [4]. The Dirac operator

$$\not{D} = \gamma^l e_l^\mu (\partial_\mu - iA_\mu - \frac{i}{2} A_\mu^{mn}\sigma_{mn}) \tag{15}$$

is hermitian for euclidean signature in the scalar product

$$(\phi, \psi) \equiv \int d^4x \sqrt{g}\phi^\dagger\psi, \tag{16}$$

but its combination with a mass term, real or complex, is not hermitian. A positive operator, needed for defining the zeta function, can then be defined as

$$\begin{aligned} \Delta \quad = \quad & [i \not{D} - m \exp(i\theta\gamma^5)]^\dagger \\ & [i \not{D} - m \exp(i\theta\gamma^5)]. \end{aligned} \tag{17}$$

It is easy to see that

$$\Delta = (\not{D})^2 + m^2. \tag{18}$$

Hence θ does not enter Δ or the zeta function

$$\zeta(s, \Delta) \equiv \mathrm{Tr}(\Delta^{-s}). \tag{19}$$

The logarithm of the fermion determinant, defined as

$$-\frac{1}{2}\zeta'(0, \Delta) - \frac{1}{2}\ln\mu^2\zeta(0, \Delta), \tag{20}$$

with some μ, is also independent of θ. Since there is no violation of parity or time-reversal at $\theta = 0$, it follows that there is no violation even for non-zero values. In other words, the parity and time-reversal symmetries found above for the classical action involving quarks with a complex fermion mass term are not anomalous and survive quantization even in curved spacetime. The γ^5 phase θ violates neither parity nor time-reversal even in gravitational interactions, just as in the strong interactions of quarks [5]. As in that case, violations can occur if there are direct $R\tilde{R}$ terms [6].

Some remarks about anomalies may be apposite here. If a fermion is coupled to a gauge field or a gravitational field, the fermion measure has to be defined with care and is not invariant under a chiral transformation of the fermion. This is the meaning of the statement that there is a chiral or axial anomaly. However, what we are considering here are time-reversal and parity transformations. We need to know whether the measure is invariant under these discrete transformations. A possibility of non-invariance arises because the discrete transformations (13,14) involve chiral transformations. If the measure is not invariant, the classical symmetries will be anomalous and the fermion determinant will not have the symmetries. The construction of the fermion determinant directly shows that it is independent of θ and therefore has time-reversal and parity symmetries. This means that the measure implicit in the zeta function regularization is invariant under (13,14) and these transformations are free from anomalies. The classical fermion action also has gauge invariance, general coordinate invariance and local Lorentz invariance. These are all maintained by the measure of Dirac fermions.

References

[1] V. Baluni, Phys. Rev. **D19** (1979) 2227

[2] H. Banerjee, D. Chatterjee and P. Mitra, Phys. Lett. **B573** (2003) 109

[3] T. Kimura, Prog. Theo. Phys. **42** (1969) 1191

[4] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications*, (World Scientific, Singapore, 1994);

[5] P. Mitra, J. Phys. **A 40** (2007) F525

[6] S. Deser, M. Duff and C. Isham, Phys. Lett. **B93** (1980) 419